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On the stability of slowly varying flow: the divergent channel

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The linear stability of a slowly varying flow, the flow in a diverging straight-walled channel, is studied using a modification of the 'WKB' or 'ray' method. It is shown that 'quasi-parallel' theory, the usual method for handling such flows, gives the formally correct lowest-order growth rate; however, this growth rate can be substantially in error if its magnitude is comparable to that of the rate of change of the basic state. The method used clearly demonstrates the dependence of the growth rate, wavenumber, neutral curves, etc., on the cross-stream variable and on the flow quantity under consideration. When applied to the divergent channel, the method yields a much wider 'unstable' region and a much lower 'critical' Reynolds number (depending on the flow quantity used) than those predicted by quasi-parallel theory. The determination of the downstream development of waves of constant frequency shows that waves of all frequencies eventually decay.

1. Introduction

Studies of the linear stability of steady shear flows that are nearly parallel, such as boundary layers, jets and wakes, have traditionally relied on a quasi-parallel assumption. Since the basic flow is changing slowly, it is argued, the stability at some point depends only on the local properties of the flow, i.e. the local profile, Reynolds number, etc. Without further justification, the mean flow is taken to be parallel and the usual analysis by normal modes follows (e.g. Schlichting 1968).

Although this approach has had great qualitative success (cf. Schlichting), recent experiments (Ross *et al.* 1970; Mattingly & Criminale 1972; Scotti & Corcos 1972) have shown systematic differences from the quasi-parallel theory. Moreover, from a theoretical viewpoint, the quasi-parallel approach is obviously deficient in two respects. First, it can only determine whether a wave is growing or decaying at a particular point; it cannot determine the solution as a function of the downstream co-ordinate. Second, it cannot take into account, or even

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estimate, the effect of the variation of the basic state on the local growth. The method of approximation used here, which we shall refer to as the slowly varying approximation, but which is perhaps better known as the WKB, ray or multiple-scaling method, overcomes these deficiencies. It provides a rational method of approximation, in which the quasi-parallel theory will appear at lowest order.

Let us be careful about our use of the terminology 'parallel' and 'quasi-parallel'. By 'parallel flow theory' we refer to the familiar linear stability theory for flows that are uniform in at least one direction and, for the purposes of this discussion, steady in time. The linear 'instability' of these flows is readily understood since the growth rate (in time or space) is constant. 'Quasi-parallel (or quasi-steady) theory' is taken to be the treatment of slowly varying flows by assuming that the local growth rate can be determined by parallel flow theory; it recognizes that the growth rate will be a function of time and/or space. If an overall change of amplitude is required, it is calculated by integrating the local growth rate, as shown, for example, in equation (39) below.

When the growth rate is not constant, the meaning of 'instability' is not so clear (Shen 1961). Waves can pass through regions of growth ('unstable' regions) and regions of decay. In some cases (as we shall see for the divergent channel), all waves eventually decay, even those which experience growth, so the flow appears to be stable on an absolute or overall basis. However, the mechanism responsible for the transfer of energy from the mean flow to a disturbance is still present. If disturbances are allowed to be completely random in space and time, then at any particular point (in space and time) some component of the resulting flow field will still be growing, even though each wave taken alone eventually decays.

There is also the problem of what to use as a measure of amplitude. Shen argues that, since the basic state is changing, the growth or decay of a disturbance should be measured relative to the basic flow. He suggests the use of a growth rate based on the ratio of disturbance kinetic energy to basic flow kinetic energy. We shall follow Shen's suggestion, adapting his definition to space-dependent flows, and shall present results for both 'relative' and 'absolute' measures of the amplitude.

A further complication for unsteady or non-parallel flow is that the various disturbance flow quantities, the stream function, velocity components, energy density, etc., need not have the same growth rate. This rather surprising situation, in comparison with parallel flow theory, has recently been demonstrated very nicely by Bouthier (1973) using essentially the same method as we use here (see §4). We shall see that different flow quantities have different neutral curves.

Finally, there is the question of nonlinear effects (Shen 1961; Rosenblat 1968). If a disturbance sustains a period of growth before decaying, its amplitude may become large enough for nonlinear effects to be important. The flow might be 'unstable' on a practical basis. Of course, the linear theory presented here cannot predict such an occurrence, but the results of Eagles (1973) on nonlinear self-interaction of waves in a diverging channel may be relevant; this is discussed in §6.

These points require further discussion than we can provide here. Our aim will be merely to illustrate them by examining a specific example of unstable, steadily varying flow.

The slowly varying approximation has been developed primarily for conservative systems which support neutral waves (e.g. Keller 1958; Lewis 1964; Bretherton 1966, 1968), but it appears to work just as well for flows that are unstable. More than ten years ago, Benney & Rosenblat (1964) suggested that the method be applied to stability problems, but this does not appear to have occurred until six years later, when Rosenblat & Herbert (1970) considered thermal convection in which the temperature of one boundary is slowly oscillating in time. (It is immaterial, as far as the method is concerned, whether the variation of the basic state is in time or space, or both.) Now Bouthier (1972, 1973) has developed the method for steady, spatially dependent shear flows. In the first paper, he describes the method in general; in the second, he applies it to the stability of the boundary layer on a flat plate. Gaster (1974) has also applied the method (with some variations in the formalism) to the boundary layer. Other contributions to the general theory have been made by Nayfeh, Mook & Saric (1974) and Drazin (1974).†

The method may be applied to flows whose rate of change in time or in the direction of wave propagation is small (but the overall change need not be small). The local properties of the wavelike disturbance are assumed to be slowly varying functions of time and/or space. For example, to lowest order the solution might be written as the real part of

$$A(X,T)f(y;X,T)\exp[i\theta(x,t)],$$

where (x,t) and (X,T) are the 'fast' and 'slow', streamwise and time co-ordinates and y is the transverse co-ordinate. The frequency $\Omega(X,T)$ and the wavenumber $K(X,T)$ are defined to be the derivatives of the 'phase function' $\theta(x,t)$:

$$\Omega \equiv -\partial\theta/\partial t, \quad K \equiv \partial\theta/\partial x.$$

They are related by a 'local' dispersion relation which, for unstable or dissipative flows, yields complex values for Ω and/or K. Thus θ is allowed to be complex, in contrast to the situation for neutral waves. The vertical structure $f(y)$ is determined by a local eigenvalue problem, parametrically dependent on X and T. The 'amplitude function' $A(X,T)$ is determined by the 'amplitude equation', which results at higher order from a solvability condition imposed on the higher-order equations. These three functions θ, f and A are each complex in general, and thus *all* contribute to the total amplitude and total phase of the wave.

For a basic state that is steady, the coefficients of the disturbance equation will be independent of time. Thus solutions having constant frequency are possible. We shall consider here waves of real frequency, waves that would be

† Unfortunately there are flaws in both of these studies. Nayfeh *et al.*, in their expressions for the 'corrected' wavenumber and growth rate, do not include the downstream variation of the eigenfunction, that is, the terms containing f in our equations (27) and (38). Drazin omits the 'amplitude function' (or any equivalent term) from his solution. Consequently his second-order solution [equation (19) of his paper] is in error and he cannot obtain higher-order corrections to the growth rate, etc.

generated by a wave maker in an experiment. The amplitude, wavenumber, vertical structure, etc., must be determined as functions of the (slowly varying) downstream variable.

The particular basic state considered in this paper is the flow in divergent straight-walled channels as given by the Jeffery–Hamel profiles (cf. Fraenkel 1962, 1963). The quasi-parallel stability of this flow was studied by Eagles (1966). In the next section, the basic flow is described and previous work summarized, and the governing equation for the disturbance is given. The application of the slowly varying approximation takes place in §3, and in the following section, after the meaning of 'growth rate' has been clarified, various results are presented: 'corrected' growth rates, neutral curves, etc. The numerical technique and checks performed on the solution are discussed in §5, and then a summary and further discussion follow.

We should also mention another method used in the study of nearly parallel flows. Lanchon & Eckhaus (1964) and Ling & Reynolds (1973) construct a local expansion of the slowly varying coefficients about some point downstream; for example, if $u_0(x,y)$ is the basic flow,

$$u_0(x,y) = u_0(x_0,y) + \left.\frac{\partial u_0}{\partial x}\right|_{(x_0,y)}(x-x_0)+\dots.$$

Their solutions also take the form of power series in $x-x_0$ (multiplied by an exponential factor) and are therefore only useful for x near x_0. As in the quasi-parallel theory, this type of analysis cannot determine the complete solution as a function of the downstream variable, but it does take into account the variation of the basic flow in determining the local growth rate. As Bouthier (1972) points out (following a useful description of each of the three different approaches to the problem: quasi-parallel, local expansion and slowly varying), this type of local solution is equivalent to a Taylor series expansion about x_0 of the solution obtained by the slowly varying approximation.

2. The divergent channel

The basic flow is the same as that considered by Eagles (1966, 1973, hereafter referred to as E1, E2): steady, two-dimensional, symmetric flow in a straight-walled divergent channel. For definiteness, we suppose that there is a wave maker at some position along the channel and that upstream from the generator the walls curve smoothly and slowly until they are parallel, as in figure 1. The width of the channel at the generator is nearly $2b$ (see figure 1), the divergence angle is 2α and the volumetric flow rate is taken to be $2M$. Following E1 and E2, we use modified polar co-ordinates

$$\xi = \alpha^{-1}\ln(\alpha r), \quad \eta = \phi/\alpha,$$

where r and ϕ are the usual polar co-ordinates (r has been non-dimensionalized by b). After non-dimensionalizing the time by b^2/M and the stream function by M, the vorticity equation in these co-ordinates is

$$\left[\nabla^2 - R e^{2\alpha\xi}\frac{\partial}{\partial t} - R\left(\frac{\partial\psi}{\partial\eta}\frac{\partial}{\partial\xi}-\frac{\partial\psi}{\partial\xi}\frac{\partial}{\partial\eta}\right)\right](e^{-2\alpha\xi}\nabla^2\psi) = 0, \tag{1}$$

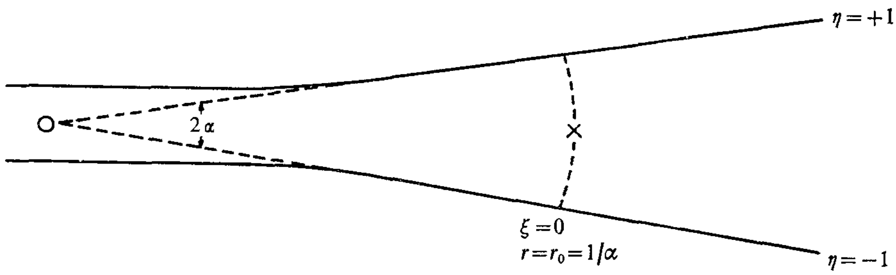

FIGURE 1. The divergent channel. ×, location of a wave maker. The length of the dashed arc at $\xi = 0$ is $2b$.

where $R = M/\nu$ is the Reynolds number and

$$\nabla^2 = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2.$$

The boundary conditions are $\psi = \pm 1$ and $\partial\psi/\partial\eta = 0$ at $\eta = \pm 1$.

As in E1 and E2 we use the Jeffery–Hamel solutions for flow in a wedge. The stream function ψ_0 of the basic flow is steady and independent of ξ, and the 'velocity profile'† $g(\eta) = \partial\psi_0/\partial\eta$ satisfies

$$g''' + 4\alpha^2 g' + 2\alpha R g g' = 0, \tag{2}$$

where a prime indicates differentiation with respect to η. We should like to study the 'more unstable' profiles within this family: the ones that have inflexion points. In the limit of small α, inflexional profiles are obtained if αR remains $O(1)$, that is, if R is large enough. In practice, if $\alpha = 0{\cdot}1$, R need only be approximately 20. Note that, as $\alpha \to 0$ with R fixed, the flow must approach plane Poiseuille flow, which we do not wish to consider in this paper. Therefore, we regard αR as fixed as $\alpha \to 0$ and obtain

$$g(\eta; R, \alpha) = w(\eta;\gamma) + O(\alpha^2), \tag{3}$$

where $$\gamma = \alpha R$$

and $$w''' + 2\gamma w w' = 0,$$

with $$w(\pm 1) = 0, \quad \int_{-1}^{+1} w\, d\eta = 2.$$

These profiles have been described in E1 and E2; two which are used in this paper are shown in figure 2. The family includes Poiseuille flow ($\gamma = 0$), flows with inflexion points ($\gamma > 1{\cdot}80$), and profiles with reversed flow near the walls ($5{\cdot}46 > \gamma > 4{\cdot}71$).

The equation for a small two-dimensional disturbance is obtained by allowing

$$\psi \to \psi_0(\eta) + \psi(\xi, \eta, t)$$

and linearizing. This yields

$$\left[R^{-1}\left\{\left(\frac{\partial}{\partial\xi} - 2\alpha\right)^2 + \frac{\partial^2}{\partial\eta^2}\right\} - e^{2\alpha\xi}\frac{\partial}{\partial t} - g(\eta)\left(\frac{\partial}{\partial\xi} - 2\alpha\right)\right]\nabla^2\psi + g''(\eta)\frac{\partial\psi}{\partial\xi} + 2\alpha g'(\eta)\frac{\partial\psi}{\partial\eta} = 0, \tag{4}$$

† The *actual* velocity in the radial direction is $u_0 = r^{-1}\partial\psi_0/\partial\phi = e^{-\alpha\xi}g(\eta)$; that is, it decays like $1/r$.

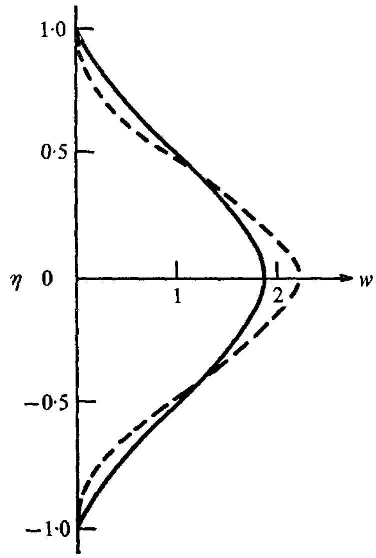

FIGURE 2. Velocity profiles $w(\eta;\gamma)$. ——, $\gamma = 3{\cdot}57$; – –, $\gamma = 4{\cdot}71$. The profile for $\gamma = 4{\cdot}09$ lies between these.

with boundary conditions

$$\psi = \partial\psi/\partial\eta = 0 \quad \text{at} \quad \eta = \pm 1. \tag{5}$$

Consider (4) when $\alpha \to 0$. Dropping the terms proportional to α and using (3), we have

$$\left[R^{-1}\nabla^2 - e^{2\alpha\xi}\frac{\partial}{\partial t} - w(\eta)\frac{\partial}{\partial \xi}\right]\nabla^2\psi + w''(\eta)\frac{\partial\psi}{\partial\xi} = 0. \tag{6}$$

(The R^{-1} term is formally $O(\alpha)$ also, but it is retained at lowest order because it contains the higher-order derivatives. This is explained further below.) Equation (6) is typical of perturbation equations for which the basic flow is nearly parallel. The coefficients depend on both the transverse variable η and the streamwise variable ξ, but the dependence on ξ is weak (for small α). The quasi-parallel theory would now ignore the variation of the coefficients with ξ, and normal-mode solutions could then be found, such as $\psi = f(\eta)\exp[i(k\xi - \omega t)]$, where $f(\eta)$ satisfies the Orr–Sommerfeld equation.

This was the approach used in E1. The transverse structure of the basic flow, the profile $w(\eta;\gamma)$, was retained, but all other effects of the divergence, including the development of the wave downstream, were neglected. Using (6) with $e^{\alpha\xi}$ set equal to unity, neutral curves were determined in the usual way, and the flows were found to be unstable for large enough Reynolds number. In the next section, we reconsider (4) using the slowly varying approximation (and the same basic flow profiles).

Before going on to the slowly varying analysis, let us consider the retention of the viscous term in (6) and in the equations to follow in the next section. Since $R^{-1} = O(\alpha)$ for the basic flows under consideration, this term is formally of higher order. However, it contains the highest-order derivatives and will in fact be dominant somewhere in the flow field. Thus it is clear that if one wishes to use numerical techniques to solve for the vertical structure, as we do, this term must be retained in the lowest-order problem. [See Lanchon & Eckhaus (1964) for an analysis using singular perturbation techniques.] A numerical approach is especially appropriate for these flows because the Reynolds number is not particularly large.

This procedure can be formalized by considering an 'extended' or 'false' problem (cf. Ling & Reynolds 1973). The basic state requires a certain relationship between α and R, namely $\alpha R = \gamma$ (γ is some fixed number for the purposes of this argument); however in the equation governing the disturbance [equation (4)], they appear as independent parameters. The basic idea is to ignore the relationship between α and R as far as the disturbance is concerned and to solve (4) as if they were independent. Thus one obtains a more general solution than is required. The relationship dictated by the basic state is returned to in the final stages of the calculation when numerical values are required for α or R. The solution will be given, therefore, as an asymptotic expansion for small α and arbitrary R. As usual when using asymptotic methods, one *trusts* that the point of interest in parameter space, $\alpha = \gamma/R$, is within the range of validity of the approximation.†

3. The slowly varying approximation

Equation (4), with boundary conditions (5), governs the disturbance. Since the coefficients are independent of time and slowly varying with ξ, we look for constant frequency solutions of the form

$$\psi(\xi, \eta, t) = \Psi(\eta, X) \exp[i\{\theta(\xi) - \beta t\}] + \text{c.c.}, \tag{7}$$

† It might be thought that the singular nature of the perturbation for $\alpha \to 0$ with $R = O(\alpha^{-1})$ would automatically invalidate this 'imbedding' procedure. However, one can show for simple model problems that this is not the case. For example, consider

$$\epsilon\phi'' + (1+\epsilon)\,\phi' = 1, \quad \phi(0) = \phi(1) = 0, \tag{A}$$

where an approximate solution for $\phi(x;\epsilon)$ is to be found for $0 < \epsilon \ll 1$. Using standard singular perturbation techniques, inner and outer expansions (which we shall not write out) can easily be found. The method used in the paper artificially separates the singular perturbation in (A) from the regular perturbation. Consider the 'false' problem

$$\lambda\psi'' + (1+\epsilon)\,\psi' = 1, \quad \psi(0) = \psi(1) = 0 \tag{B}$$

for $\psi(x;\lambda,\epsilon)$. For $0 < \epsilon \ll 1$ and λ fixed, an asymptotic expansion is developed:

$$\psi(x;\lambda,\epsilon) \sim \psi^{(1)}(x;\lambda) + \epsilon\psi^{(2)}(x;\lambda) + \dots. \tag{C}$$

We shall not write this out explicitly, but note that terms like $\exp(-x/\lambda)$ and $\exp(-1/\lambda)$ appear. If (C) is now evaluated at our point of interest in parameter space, $\lambda = \epsilon$, we obtain

$$\psi(x;\epsilon,\epsilon) \sim \psi^{(1)}(x;\epsilon) + \epsilon\psi^{(2)}(x;\epsilon) + \dots. \tag{D}$$

It can be shown that (D) is a proper asymptotic solution to (A) in the sense that the difference between the exact solution and a partial sum of N terms is $O(\epsilon^N)$ uniformly in x (at least for $N \leqslant 3$). If (D) is re-expanded in the two regions x fixed and $x = O(\epsilon)$, the result is the same inner and outer expansions as were obtained by treating (A) directly. It could be argued that (D) is a more accurate representation of the exact solution because it retains the transcendentally small terms such as $\exp(-1/\epsilon)$.

where c.c. refers to the complex conjugate of the preceding terms and $X = \alpha\xi$ is the slow variable. The complex phase function $\theta(\xi)$, yet to be determined, describes the fast variation, but its derivative, the wavenumber, is assumed to be slowly varying. That is,

$$K \equiv d\theta/d\xi = K(X),$$

and therefore
$$\theta(\xi) = \int^{\xi} K(\alpha\xi)\,d\xi = \alpha^{-1}\Theta(X), \tag{8}$$

where
$$\Theta(X) \equiv \int^{X} K(X)\,dX.$$

Note that K is the wavenumber in ξ space. In r space the wavenumber would be

$$\kappa \equiv d\theta/dr = (d\xi/dr)(d\theta/d\xi) = e^{-X}K(X).$$

Substituting the trial solution (7) into (4) yields the equation for Ψ:

$$\begin{aligned} &R^{-1}(D^2-K^2)^2\Psi + i(\beta e^{2X} - wK)(D^2-K^2)\Psi + iKw_{\eta\eta}\Psi \\ &\quad + \alpha[2iK_X R^{-1}(D^2-3K^2)\Psi - 4iKR^{-1}(D^2-K^2)(\Psi-\Psi_X) \\ &\quad + i\beta e^{2X}(iK_X\Psi + 2iK\Psi_X) + 3wKK_X\Psi + 2w(D^2-K^2)\Psi \\ &\quad - w(D^2-3K^2)\Psi_X + 2w_\eta D\Psi + w_{\eta\eta}\Psi_X] = O(\alpha^2), \end{aligned} \tag{9}$$

where $w = w(\eta;\gamma)$ is the basic flow profile to $O(\alpha)$. Subscripts and $D \equiv \partial/\partial\eta$ indicate differentiation. The boundary conditions are

$$\Psi = D\Psi = 0 \quad \text{at} \quad \eta = \pm 1. \tag{10}$$

We now treat R and γ as constant and look for a solution as $\alpha \to 0$, as explained in the previous section. Thus, we let

$$\Psi(\eta, X; R, \gamma, \alpha) = \Psi_1(\eta, X; R, \gamma) + \alpha\Psi_2(\eta, X; R, \gamma) + \dots$$

and obtain the following sequence of problems.

At $O(1)$,

$$L\Psi_1 \equiv [R^{-1}(D^2-K^2)^2 + i(\beta_I - wK)(D^2-K^2) + iKw_{\eta\eta}]\Psi_1 = 0, \tag{11a}$$

$$\Psi_1 = D\Psi_1 = 0 \quad \text{at} \quad \eta = \pm 1, \tag{11b}$$

where $\beta_I \equiv \beta e^{2X} = \beta_I(X)$ and $K = K(X)$. This is the standard Orr–Sommerfeld problem except that where the frequency and wavenumber would appear we now have functions of the slow variable. (We shall call β_I the 'intrinsic' frequency; it is the value that the frequency would take if the non-dimensionalization were carried out at the position X downstream.) Since X appears only parametrically, a solution of the form

$$\Psi_1(\eta, X) = A(X) f(\eta; \beta_I(X), K(X)) \tag{12}$$

may be found, where $f(\eta; \beta_I, K)$ is a solution of (11) for fixed X and is normalized in some specified (and arbitrary) manner. Figure 3 shows the results of two calculations of f for two values of β_I. For all our calculations the normalization chosen was $f(0) = 1$ for all X values. Therefore $A(X)$, the (complex) amplitude function, which is yet to be determined, may be interpreted as the amplitude of the stream function on the centre-line of the channel. It is not the 'total' amplitude (see §4), and it is not a measure of amplitude for other flow quantities.

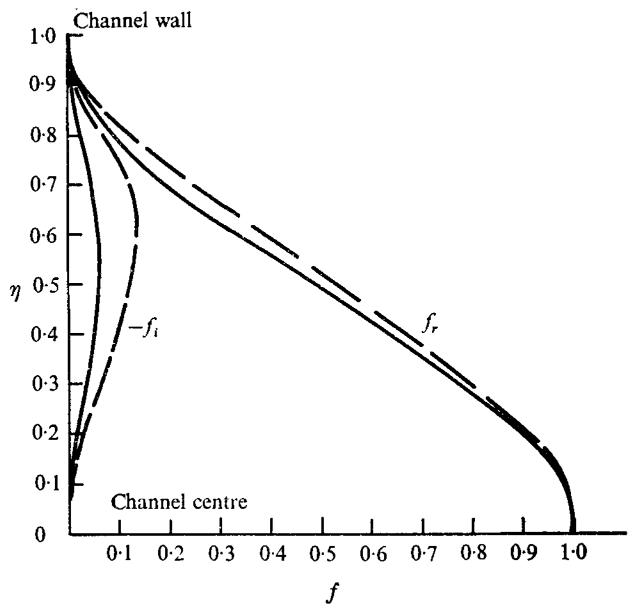

FIGURE 3. Real and imaginary parts of the eigenfunction $f = f_r + if_i$ for $\gamma = 4{\cdot}09$ and $R = 45$. ——, $\beta_I = 0{\cdot}445$; – –, $\beta_I = 2{\cdot}205$. These values of β_I correspond to points A and B in figures 4 and 5 (*b*).

Of course, in order to obtain such a solution to this eigenvalue problem, a certain characteristic relation must hold; for example,

$$K = F(\beta_I, R, \gamma).$$

Thus, K as a function of X is given by

$$K(X) = F(\beta e^{2X}, R, \gamma), \tag{13}$$

and $\theta(\xi)$ itself may be determined from (8). For β, R and γ real, $K(X)$ [and therefore $\theta(\xi)$] will be complex in general. It is understood that we are considering the 'most unstable' mode, the one with the most negative value of the imaginary part of K for each value of β and X.

At $O(\alpha)$,

$$L\Psi_2 = L_1 \partial\Psi_1/\partial X + K_X L_2 \Psi_1 + L_3 \Psi_1, \tag{14a}$$

$$\Psi_2 = D\Psi_2 = 0 \quad \text{at} \quad \eta = \pm 1, \tag{14b}$$

where L is the Orr–Sommerfeld operator defined in (11*a*) and L_1, L_2 and L_3 are also differential operators with respect to η:

$$\left.\begin{aligned} L_1 &\equiv -4iKR^{-1}(D^2-K^2) + 2K\beta_I + w(D^2-3K^2) - w_{\eta\eta}, \\ L_2 &\equiv -2iR^{-1}(D^2-3K^2) + \beta_I - 3wK, \\ L_3 &\equiv 4iKR^{-1}(D^2-K^2) - 2w(D^2-K^2) - 2w_\eta D. \end{aligned}\right\} \tag{15}$$

The first two inhomogeneous terms of (14*a*) are seen to arise from the particular form of the solution (7); however, the third comes directly from the terms proportional to α in (4).

We note in passing that the operators L, L_1 and L_2 are related in a special way. If we consider L to be a function, $L = L(D, K, \beta_I, R, \gamma, \eta)$, we can evaluate the derivatives of L with respect to these parameters. One may easily verify that

$$L_1 = i\,\partial L/\partial K, \quad L_2 = \tfrac{1}{2}i\,\partial^2 L/\partial K^2. \tag{16), (17}$$

This can be shown to be a general result, valid for arbitrary differential equations such as (4).

Let us rewrite problem (14) using (12):

$$\left.\begin{aligned} L\Psi_2 = (L_1 f)\frac{dA}{dX} + \left(L_1 \frac{\partial f}{\partial X}\right) A + K_X (L_2 f) A + (L_3 f) A, \\ \Psi_2 = D\Psi_2 = 0 \quad \text{at} \quad \eta = \pm 1. \end{aligned}\right\} \tag{18}$$

(f is now taken to be a function of η and X, i.e. $f = f(\eta, X)$.) This inhomogeneous system can have a solution only if a certain solvability condition is satisfied. Let $\tilde{L}$ be the operator adjoint to L,

$$\tilde{L} \equiv R^{-1}(D^2 - K^2)^2 + i(\beta_I - wK)(D^2 - K^2) - 2iKw'D,$$

and $\tilde{f}$ be a solution to

$$\left.\begin{aligned} \tilde{L}\tilde{f} = 0, \\ \tilde{f} = D\tilde{f} = 0 \quad \text{at} \quad \eta = \pm 1. \end{aligned}\right\} \tag{19}$$

Then, provided that Ψ_2 satisfies the boundary conditions, it can be shown that

$$\int_{-1}^{+1} \tilde{f} L \Psi_2 \, d\eta = 0,$$

which implies that

$$C_1(X)\, dA/dX + C_2(X) A + C_3(X) A + C_4(X) A = 0, \tag{20}$$

where

$$\left.\begin{aligned} C_1(X) = \int_{-1}^{+1} \tilde{f} L_1 f \, d\eta, \quad C_2(X) = \int_{-1}^{+1} \tilde{f} L_1 f_X \, d\eta, \\ C_3(X) = K_X \int_{-1}^{+1} \tilde{f} L_2 f \, d\eta, \quad C_4(X) = \int_{-1}^{+1} \tilde{f} L_3 f \, d\eta. \end{aligned}\right\} \tag{21}$$

Equation (20) is the required amplitude equation. It may be written as

$$dA/dX + H(X) A(X) = 0 \tag{22}$$

and easily integrated provided that $C_1(X)$ does not vanish (which is the case in our examples).

The solution then takes the form

$$\begin{aligned} \Psi = A(X) f(\eta, X) \exp\{i(\theta(\xi) - \beta t)\} + \text{c.c.} \\ + \alpha[\Psi_2(\eta, X) \exp\{i(\theta(\xi) - \beta t)\} + \text{c.c.}] + O(\alpha^2), \end{aligned} \tag{23}$$

in which the components of the lowest-order solution are known. The next-order correction, Ψ_2, could be calculated from (18). (Ψ_2 would contain a homogeneous solution of the form $A_2(X) f(\eta, X)$; $A_2(0)$ can be specified by requiring that Ψ_2 be orthogonal to the lowest-order solution at $X = 0$ (cf. Ling & Reynolds 1973), but $A_2(X)$ for $X > 0$ can only be determined by going to the next order in the calculation, where another amplitude equation would arise.) However, to obtain the $O(\alpha)$ correction to the growth rate, it is not necessary to calculate the $O(\alpha)$ correction to the vertical structure. This is demonstrated in the next section.

4. The growth rates, neutral curves and downstream development

The wave amplitude that would be observed in an experiment, i.e. the 'total' or 'physical' amplitude, is, in terms of the stream function,

$$\operatorname{amp}\psi \equiv 2|\Psi \exp[i(\theta-\beta t)]| = 2|\Psi| \exp(-\theta_i), \tag{24}$$

where θ_i is the imaginary part of θ. We define a growth rate based on ψ in ξ space as

$$G_\xi(\psi) \equiv (\operatorname{amp}\psi)^{-1}\,\partial(\operatorname{amp}\psi)/\partial\xi. \tag{25}$$

(The growth rate in r space would be

$$G_r(\psi) \equiv (\operatorname{amp}\psi)^{-1}\,\partial(\operatorname{amp}\psi)/\partial r$$

and thus differs by a positive factor since $\partial/\partial r = e^{-X}\,\partial/\partial\xi$.) Using (24), (25) becomes

$$G_\xi(\psi) = -K_i + \alpha|\Psi|_X/|\Psi|, \tag{26}$$

where $K_i = d\Theta_i/d\xi$ is the imaginary part of the wavenumber. The expansion for Ψ,

$$\Psi(\eta, X) = A(X)f(\eta, X) + \alpha\Psi_2(\eta, X) + \ldots,$$

may be substituted, yielding

$$G_\xi(\psi) = -K_i + \alpha(|A|_X/|A| + |f|_X/|f|) \tag{27}$$

to $O(\alpha)$.

We see that formally the growth rate is the same to lowest order as that given by the quasi-parallel theory: $-K_i(X)$. However, in the unstable region, according to that theory, the magnitude of K_i is small and is likely to be comparable to the value of α. Furthermore when K_i is equal to zero, i.e. at the neutral points determined by quasi-parallel theory, there is still growth or decay due to the higher-order effects. Thus, the higher-order corrections are essential in determining the growth through the unstable region and in determining the correct neutral points, the values of X where the growth rate is truly zero.

Before drawing 'corrected' neutral curves, we must be quite careful about the growth rate as defined in (25). The $O(\alpha)$ corrections exhibit two new, and perhaps surprising, features. First, the growth rate varies across the channel owing to the dependence of f upon X. And second, the growth rate is different for different flow quantities. Consider the velocity components, given in polar co-ordinates by

$$\left.\begin{aligned} u &= \frac{1}{r}\frac{\partial\psi}{\partial\phi} = e^{-X}\frac{\partial\psi}{\partial\eta} = e^{-X}\left[Af_\eta \exp\{i(\theta-\beta t)\} + \text{c.c.}\right] + O(\alpha), \\ v &= -\frac{\partial\psi}{\partial r} = -e^{-X}\frac{\partial\psi}{\partial\xi} = -e^{-X}\left[iKAf \exp\{i(\theta-\beta t)\} + \text{c.c.}\right] + O(\alpha). \end{aligned}\right\} \tag{28}$$

The physical amplitudes of these quantities are

$$\left.\begin{aligned} \operatorname{amp} u &= 2e^{-X}\,|A|\,|f_\eta|\exp(-\theta_i)+O(\alpha), \\ \operatorname{amp} v &= 2e^{-X}\,|K|\,|A|\,|f|\exp(-\theta_i)+O(\alpha) \end{aligned}\right\} \tag{29}$$

and therefore

$$\left.\begin{aligned} G_\xi(u) &= -K_i+\alpha\left(-1+\frac{|A|_X}{|A|}+\frac{|f_\eta|_X}{|f_\eta|}\right), \\ G_\xi(v) &= -K_i+\alpha\left(-1+\frac{|A|_X}{|A|}+\frac{|f|_X}{|f|}+\frac{|K|_X}{|K|}\right) \end{aligned}\right\} \tag{30}$$

to $O(\alpha)$. These growth rates can differ considerably from the growth rate of the stream function. In fact, in one example we found that $G_\xi(u)$ evaluated at $\eta = \pm 0{\cdot}5$ remained negative at all X, even in the region where $-K_i$ was positive.

Thus the question arises as to what the 'proper' measure is for the growth of the wave. Of course (23) may be regarded simply as the description of the perturbation flow field, and any particular aspect of the flow may be derived from it. If one has experimental data to compare with the calculation, one obviously must use the same quantity as that which was observed. Unfortunately, we do not know of any suitable observations of flow in a divergent channel, and so, in order to compare with the quasi-parallel theory, we need a general measure of the strength of the disturbance as it develops downstream. We decided to use a mean kinetic energy density, averaged over time and integrated across the channel, defined as follows:

$$E \equiv \frac{1}{2}\int_{-\alpha}^{+\alpha}(\overline{u^2}+\overline{v^2})\,r\,d\phi, \tag{31}$$

where an overbar indicates an average over a period. Following Shen (1961) we also defined a 'relative' energy density

$$\hat{E} \equiv E/E_0, \tag{32}$$

where E_0 is the kinetic energy density of the basic flow:

$$E_0 = \frac{1}{2}\int_{-\alpha}^{+\alpha} u_0^2\,r\,d\phi = \frac{1}{2r}\int_{-1}^{+1}[g(\eta)]^2\,d\eta = e^{-X}\,\Gamma, \tag{33}$$

where $$\Gamma \equiv \frac{1}{2}\int_{-1}^{+1}[g(\eta)]^2\,d\eta = \frac{1}{2}\int_{-1}^{+1}[w(\eta)]^2\,d\eta = \Gamma(\gamma) \tag{34}$$

to $O(\alpha)$. Using (28) and changing the co-ordinates, (31) and (32) yield

$$\left.\begin{aligned} E &= e^{-X}\,|A|^2e^{-2\theta_i}\int_{-1}^{+1}(|f_\eta|^2+|K|^2\,|f|^2)\,d\eta+O(\alpha), \\ \hat{E} &= \Gamma^{-1}|A|^2e^{-2\theta_i}\int_{-1}^{+1}(|f_\eta|^2+|K|^2\,|f|^2)\,d\eta+O(\alpha). \end{aligned}\right\} \tag{35}$$

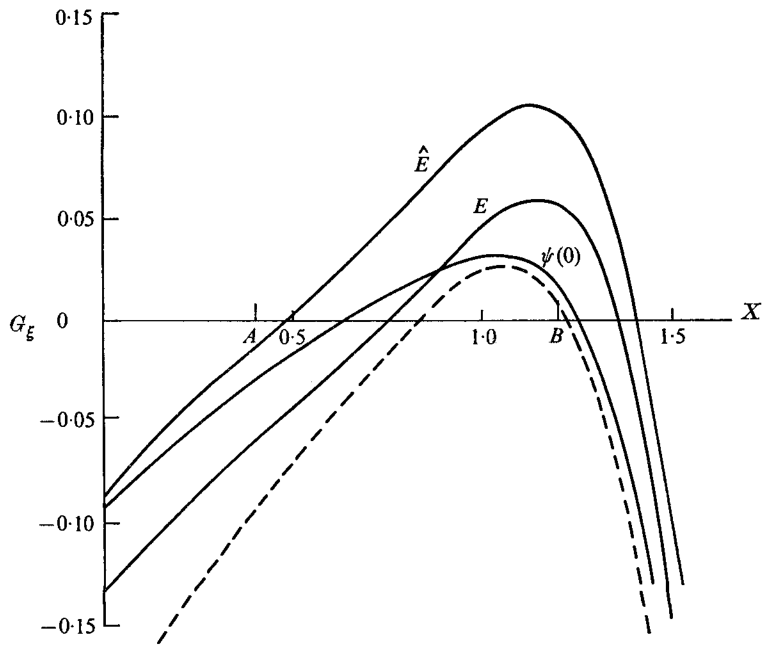

FIGURE 4. Growth rates as a function of X for $\gamma = 4{\cdot}09$, $R = 45$ and $\beta = 0{\cdot}2$. – –, quasi-parallel growth rate $-K_i$. The eigenfunctions at points A and B are shown in figure 3.

For growth rates based on these quantities, we must include a factor of a half in the definition to enable comparison with the other growth rates; that is,

$$\left.\begin{aligned} G_\xi(E) &\equiv \tfrac{1}{2}E^{-1}dE/d\xi = -K_i + \alpha\left[-\frac{1}{2}+\frac{|A|_X}{|A|}+\frac{P_X}{2P}\right]+O(\alpha^2),\\ G_\xi(\hat{E}) &\equiv \tfrac{1}{2}\hat{E}^{-1}d\hat{E}/d\xi = -K_i + \alpha\left[\frac{|A|_X}{|A|}+\frac{P_X}{2P}\right]+O(\alpha^2), \end{aligned}\right\} \tag{36}$$

where
$$P(X) \equiv \int_{-1}^{+1} (|f_\eta|^2 + |K|^2\,|f|^2)\,d\eta.$$

Figure 4 shows a typical calculation of various growth rates to $O(\alpha)$ as functions of X.† Displayed are the growth rates for E, $\hat{E}$ and ψ (evaluated at $\eta = 0$). We see that in the 'worst' case, that for $\hat{E}$, the maximum growth rate is more than three times that given by quasi-parallel theory. The dependence of the growth rates is actually upon the intrinsic frequency $\beta_I = \beta e^{2X}$, so the same curves can be applied to waves of other frequencies by a suitable shift of origin.

From a series of calculations such as this for other values of R, the neutral points (i.e. the values of X where the growth rates are zero) are easily determined as a function of the Reynolds number. This leads to the sets of neutral curves given in figure 5. Three different values of γ, corresponding to three different

† It is at this stage, when a specific value of α is required, that the relationship $\alpha R = \gamma$ is used (see the discussion at the end of §2). In practice γ and R were chosen first, and then $\alpha = \gamma/R$ was determined. As seen in figure 6, α varied from about 0·05 to 0·4.

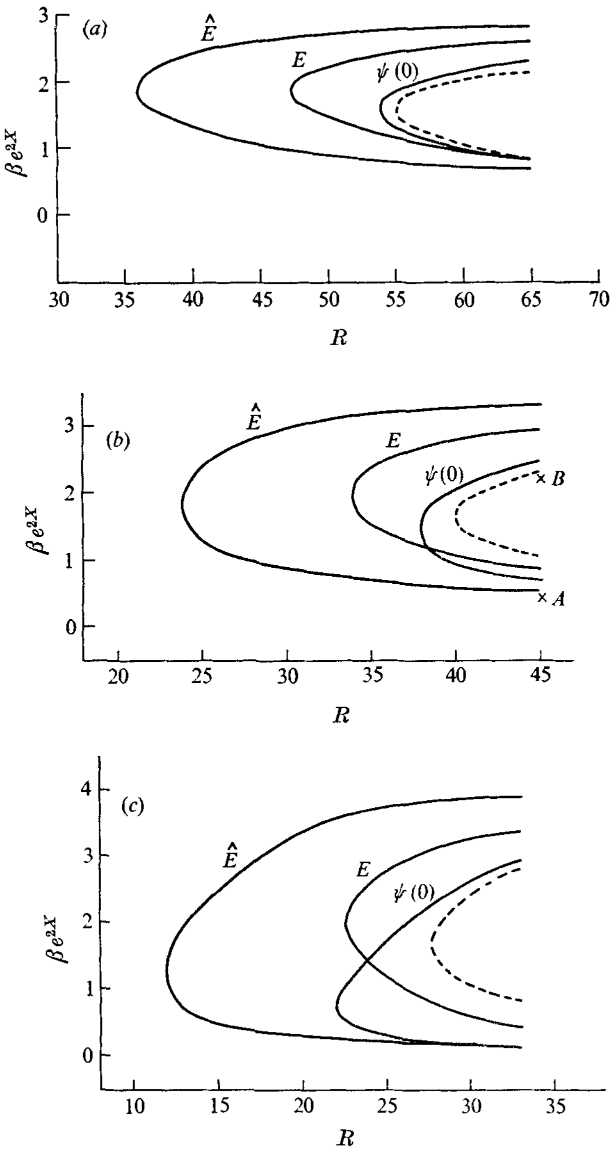

FIGURE 5. Neutral curves for three different profiles. – –, neutral curves according to quasi-parallel theory. Inside the curves there is growth; outside, decay. (a) $\gamma = 3{\cdot}57$. (b) $\gamma = 4{\cdot}09$. (c) $\gamma = 4{\cdot}71$.

flow profiles (see figure 2), have been used. The dashed curves are the neutral curves that would be determined by quasi-parallel theory. The other curves indicate the regions of growth for the various quantities. Note that the ordinate is $\beta_I = \beta e^{2X}$. For fixed β, the evolution of a wave as it travels downstream is given along a line of constant R. The waves which pass through a region of growth ultimately decay. On the other hand, for a given value of X, figure 5 indicates which frequencies correspond to positive growth (of some flow quantity).

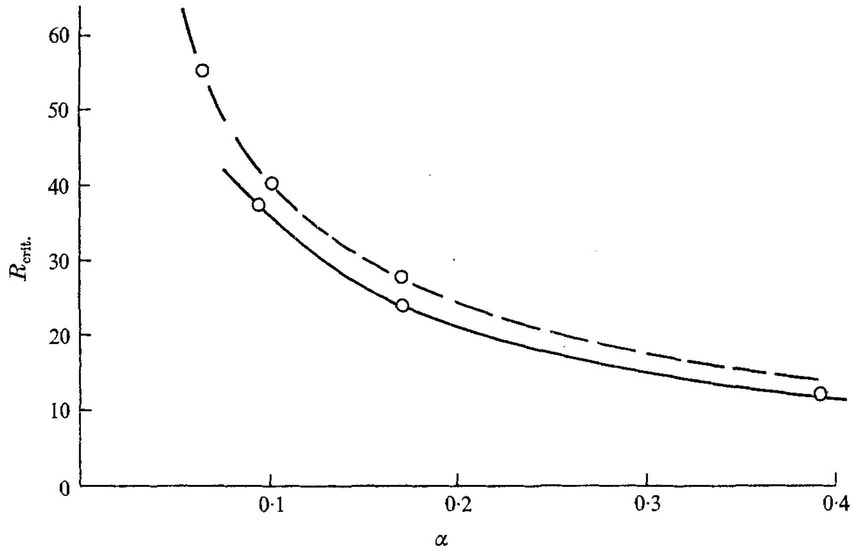

FIGURE 6. The 'critical' Reynolds number based on the relative energy $\hat{E}$. – –, quasi-parallel theory (part of this curve is from E1).

As long as the Reynolds number is larger than some 'critical' value (which depends upon the flow quantity), there is a band of growing waves for any value of X. However, this might not indicate instability. Low frequency waves will pass through a long period of decay before reaching their region of growth at large X values; the growth in the 'unstable' region cannot make up for the decay. This is demonstrated below in figure 9.

For each value of γ, a critical Reynolds number can be defined for each flow quantity, indicating when that particular flow quantity will be able to grow. Although it does not indicate whether the flow is stable or unstable, we have plotted in figure 6 the critical Reynolds number for the relative disturbance energy $\hat{E}$ in order to compare it with the one determined by quasi-parallel theory. For fixed α, the decrease in the critical Reynolds number is roughly 17 %.

In addition to changes in the growth rate at higher order, the solution (7) exhibits higher-order corrections to the wavenumber. Let us write (7) as

$$\psi = 2|\Psi|e^{-\theta_i}\cos\{\text{ph}\,\psi\}, \tag{37}$$

where
$$\text{ph}\,\psi \equiv \theta_r + \arg\Psi - \beta t$$

is the total phase of ψ (θ_r is the real part of θ and $\arg\Psi$ is the argument of Ψ). Using the expansion for Ψ, we have

$$\arg\Psi = \arg A + \arg f + O(\alpha).$$

The observed, or 'physical', wavenumber is then the derivative of the total phase; in ξ space, in terms of ψ, it is

$$N_\xi(\psi) \equiv \partial(\text{ph}\,\psi)/\partial\xi = K_r + \alpha[(\arg A)_X + (\arg f)_X] + O(\alpha^2), \tag{38}$$

where K_r is the real part of K.

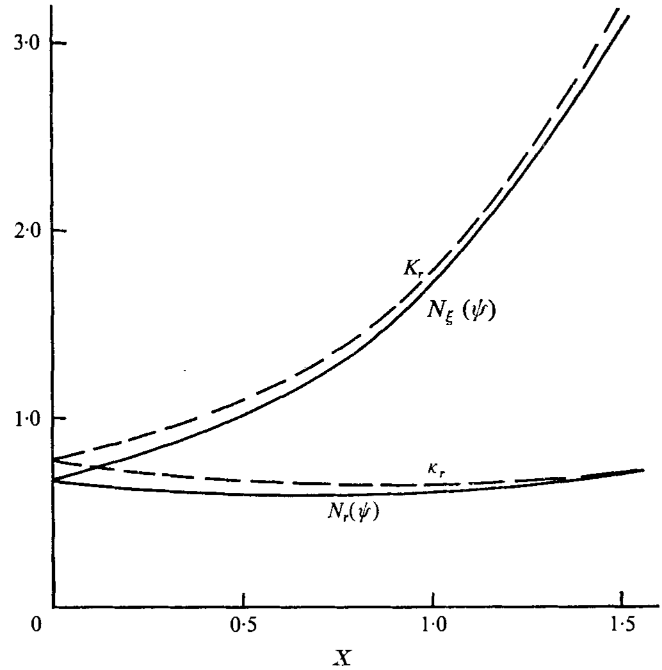

FIGURE 7. Physical wavenumbers based on ψ ($\eta = 0$) as functions of X for $\gamma = 4{\cdot}09$, $R = 45$ and $\beta = 0{\cdot}2$. – –, quasi-parallel theory.

The physical wavenumber also depends upon η and the flow quantity under consideration. For example, for the velocity components [see (28)],

$$N_\xi(u) = K_r + \alpha[(\arg A)_X + (\arg f_\eta)_X] + O(\alpha^2),$$

$$N_\xi(v) = K_r + \alpha[(\arg A)_X + (\arg f)_X + (\arg K)_X] + O(\alpha^2).$$

However, in contrast to the situation for the growth rates, the lowest-order term, i.e. the wavenumber as usually determined from quasi-parallel theory, is much larger than α and is truly dominant. This is illustrated in figure 7, where the quasi-parallel wavenumbers K_r and κ_r and the total wavenumbers $N_\xi(\psi)$ and $N_r(\psi)$ (evaluated at $\eta = 0$) are shown to be nearly equal. We note that the wavenumber that would actually be measured in r space,

$$N_r(\psi) \equiv \partial(\mathrm{ph}\,\psi)/\partial r = e^{-X} N_\xi(\psi),$$

does not change very much as the wave progresses downstream, as can also be seen in figure 8.

Figures 8 and 9 show the typical downstream development of waves. In figure 8, a wave as it would appear at some instant of time is displayed in terms of the vertical velocity measured on the centre-line of the channel. For this choice of frequency, the wave decays initially (i.e. at $X = 0$, $r = r_0 = 1/\alpha$), grows through the 'unstable' region and ultimately decays. For comparison, also plotted is the amplitude of the relative velocity, $\hat{v} \equiv v/u_0$, and the (absolute) amplitude that would be determined using quasi-parallel theory:

$$(\mathrm{amp}\, v)_{\mathrm{QP}} \equiv v_0 \exp\left\{-\frac{1}{\alpha}\int_0^X K_i\, dX\right\}, \tag{39}$$

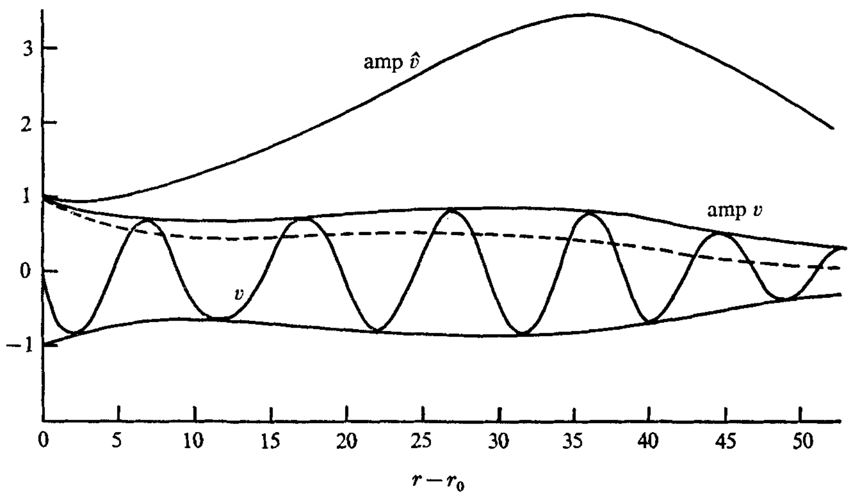

FIGURE 8. The appearance of the wave in physical space (i.e. r space) in terms of the vertical velocity on the centre-line of the channel for $\gamma = 4{\cdot}09$, $R = 45$ and $\beta = 0{\cdot}2$. – –, amplitude that would be determined by quasi-parallel theory, see (39).

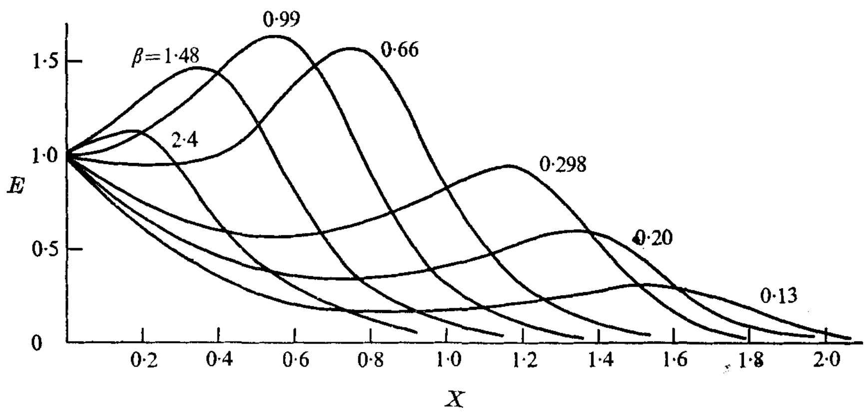

FIGURE 9. The amplitude in terms of the absolute energy density as a function of X for various frequencies; $\gamma = 4{\cdot}09$ and $R = 45$.

where v_0 = constant = 1 in figure 8. In figure 9 the amplitude in terms of the absolute energy density is shown for waves of different frequencies. Because of the dependence of the growth rates upon the intrinsic frequency, rather than on β or X separately, it can be shown that any of these curves can be obtained from any other one by a suitable shift of the origin and rescaling.

5. Numerical methods and checks

The first computational problem was to solve for the complex eigenvalue $K(X)$ for given real values of β, R, γ and X in the Orr–Sommerfeld problem (11). Minor changes in some well-tested computer programs (E2) enabled us to do this and to compute the eigenfunction f and the adjoint eigenfunction $\tilde{f}$.

It is well known that severe computational difficulties arise if $|KR|$ is large, but here we restricted ourselves to cases where $|KR| < 300$ and had no difficulty in obtaining solutions correct to about four significant figures in f.

We increased X by small steps of length h and stored the results for $K(X), f$ and $\tilde{f}$. We then obtained K_X, f_X, etc., by numerical differentiation using central difference formulae with truncation error $O(h^4)$. We were then able to evaluate the integrals required for $H(X)$ and to calculate $A(X)$ by integrating (22).

The integration of the differential equations was done by a fourth-order Runge–Kutta process, and Simpson's rule was used for all integrations. By using the symmetry of the disturbance stream function, we were able to calculate all the results over the range (0, 1) in η. We used both 40 and 20 steps in η over this range, and experimented with various sizes of the X-steps. Most of the calculations were done with 20 η-steps and with an X-step length of 0·04. We estimated that our results for $K(X)$ and $H(X)$ were correct to better than three significant figures in general.

We now list various checks made on the calculations.

(i) Tests on $K(X), f(\eta, X)$ and $\tilde{f}(\eta, X)$ were the same as those described in E2.

(ii) The numerical results for the profiles $w(\eta;\gamma)$ were checked against earlier results of Fraenkel (1963).

(iii) The coefficients $C_j(X)$ in (20) were calculated from expressions different from those given in (21). If we consider the operator L, which is displayed in (11a), to be a function of $K(X)$ and $\beta_I(X)$ then by taking the total derivative with respect to X of the equation $Lf = 0$ we find that f_X satisfies the equation

$$Lf_X = -\left(\frac{dL}{dX}\right)f = -\left(\frac{\partial L}{\partial \beta_I}\beta_{IX} + \frac{\partial L}{\partial K}K_X\right)f \tag{40}$$

with the same boundary conditions as for f. On using the solvability condition it follows that

$$K_X\int_{-1}^{+1}\tilde{f}\frac{\partial L}{\partial K}f\,d\eta + \beta_{IX}\int_{-1}^{+1}\tilde{f}\frac{\partial L}{\partial \beta_I}f\,d\eta = 0 \tag{41}$$

and then from (41), (21) and (16) that

$$C_1(X) = i\int_{-1}^{+1}\tilde{f}\frac{\partial L}{\partial K}f\,d\eta = -i\frac{\beta_{IX}}{K_X}\int_{-1}^{+1}\tilde{f}\frac{\partial L}{\partial \beta_I}f\,d\eta. \tag{42}$$

Using the expression for L from (11a), we have

$$\frac{\beta_{IX}}{K_X}\frac{\partial L}{\partial \beta_I} = \frac{2\beta e^{2X}}{K_X}\,i(D^2 - K^2),$$

so that the second integral in (42) is entirely different in form from the first. This provided a useful check on the numerics. Another, more complicated identity involving C_1, C_2 and C_3 can be obtained by taking the second derivative with respect to X of $Lf = 0$, and this was used to provide a further check on the original calculations.

(iv) If we know $K(X), f(\eta, X)$ and $\tilde{f}(\eta, X)$ then (41) determines $K_X(X)$. This was used as a check on our numerical differentiation of $K(X)$. We then solved (40) to provide a check on f_X.

6. Summary

We have obtained approximate solutions for waves of constant, real frequency as they travel down a slowly diverging channel. The solutions take into account the downstream development of the basic flow and of the waves; they are quite different, both quantitatively and qualitatively, from those obtained for parallel flows. We may summarize the differences as follows.

(i) The growth rate (spatial) is a function of the downstream co-ordinate. Thus, waves may pass through regions of growth and regions of decay. For the divergent channel, we have found that waves of all frequencies, even those which pass through an 'unstable' region, eventually decay as they travel far downstream from the wave maker. This result could have been obtained by use of the usual quasi-parallel theory, but the higher-order corrections that we have calculated here show that, in general, the unstable region is considerably widened and the critical Reynolds number is correspondingly reduced. (This may be a general feature of flows that are decelerating; cf. Shen 1961).

(ii) The growth rate is a function of the cross-stream variable. This occurs because the vertical structure of the disturbance, the eigenfunction, evolves as the wave travels downstream. This behaviour suggests the use of a quantity that is integrated across the channel, such as the kinetic energy density [(31) or (32)], as a general measure of the growth of the disturbance.

(iii) The growth rate is a function of the flow quantity involved, i.e. the stream function, velocity components, kinetic energy, etc. This is perhaps the most striking feature of these solutions as compared with parallel flow theory. The stream function, for example, may be growing at some point in the channel while the velocity components are decaying. This leads to different neutral curves and different critical Reynolds numbers for different flow quantities.

(iv) The wavelength is also a function of the distance downstream, the cross-stream variable and the flow quantity. However, we have found for the divergent channel that, in contrast to the results for the growth rate, the latter two effects are not very important. The lowest-order term (the wavelength as determined by quasi-parallel theory) is truly dominant.

(v) A distinction must be made between flow quantities measured relative to the basic state and those measured absolutely. Since the basic state is evolving as the wave is growing, a relative measure may be more important for determining nonlinear effects.

The slowly varying approximation that we have used is a formal perturbation scheme in which the quasi-parallel theory emerges at lowest order. Thus quasi-parallel theory is *formally* justified as giving a first approximation. However, in *practical* terms, we have seen that the quasi-parallel predictions for the growth and downstream development of the amplitude can be seriously in error. As (27), (30) and (36) show, the quasi-parallel growth rate $-K_i$ is a good approximation to the total growth rate only if it is much larger (in numerical magnitude) than the rate of variation of the basic flow (α, in our case). If the characteristic scale of the instability (according to quasi-parallel theory) is comparable to the slow scale of the basic state, the quasi-parallel theory should not be used for the growth rates or amplitude development. (The ratio $\alpha/|K_i|$ may be a useful parameter; if it is very small, the effect of the variation of the basic state on the growth rate can be safely ignored.)

The shifts in the neutral curves, figure 5, are quite large, larger than might be expected in a higher-order theory. Again, this is due to the smallness of the lowest-order term for the growth rate, and it does not indicate a failure in the method [the $O(\alpha)$ terms remain $O(\alpha)$ in numerical magnitude]. Of course, further corrections may shift the neutral curves once more, but these shifts are expected to be small [$O(\alpha)$].

We have found that waves of all frequencies eventually decay as they travel downstream. However, in the unstable region, waves may become large enough for nonlinear effects to be important. Our linear results show that unless the flow is substantially supercritical, a wave will simply not grow very much before it starts to decay. In figure 8, at a Reynolds number 12·5 % above the quasi-parallel critical value, the overall growth through the unstable region is only 30 % in terms of the absolute vertical velocity and 250 % in terms of the relative vertical velocity. In terms of the kinetic energy density, the overall growth is about 60 % for the absolute quantity (see figure 9) and about 250 % for the relative quantity (not illustrated).

Even if the amplitude becomes large enough for nonlinear effects to be important, the results of Eagles (1973) indicate that the nonlinearity will be stabilizing. His calculations are for parallel flow having the profiles considered here, but the same qualitative behaviour is expected in the slowly varying situation. Thus even waves of finite (but small) amplitude may be expected to pass through a region of growth but then to decay.

We may compare our results with those of Bouthier (1973) for the laminar boundary layer on a flat plate, in which the quasi-parallel growth rate is also small. Using essentially the same method of analysis, he also obtains substantial destabilizing shifts in the neutral curve (35 % in the critical Reynolds number, 85 % in the critical frequency), shifts which bring linear stability theory in line with experimental evidence. (Unfortunately, Bouthier is not precise about whether the flow quantity used to define the corrected neutral curve was the same as that observed in the experiments; cf. Gaster 1974.) Our results for the physical or total wavelength also compare favourably: the correction due to the variation of the basic flow increases the wavelength by only 1 or 2 % over that given by quasi-parallel theory.

The steady-state, purely oscillatory waves that we have obtained are those that one would expect to find downstream of a generator in an experiment. Unfortunately, we know of no experimental data with which to compare our results, and we do not know theoretically whether these steady-state solutions would actually exist. For this, one should solve an initial-value problem in which the wave maker is 'turned on' at some instant of time, as Briggs (1964) and Gaster (1965) have done for parallel flows. They found that, if the real part of the group velocity is positive for the unstable modes, then a transient disturbance will propagate downstream, leaving a steady-state wave behind. Even though the meaning of group velocity is not entirely clear for unstable, slowly varying flows,† it is reasonable that the same criteria would hold. For the solutions that we have investigated the real part of the group velocity is positive for all frequencies so we expect that a steady-state wave will be established.

Another aspect of the problem requiring further study is the possibility of reflexions. In ordinary WKB theory (e.g. Heading 1962), the existence of transition points implies that a reflected wave is necessary to complete the approximation to the exact solution. If the transition point is on the real axis, there is total reflexion; the reflected wave has an $O(1)$ amplitude. If the transition point is off the real axis, there is partial or weak reflexion; the amplitude of the reflected wave is typically $O(e^{-l/\alpha})$, where α is the small parameter and l is proportional to the distance from the real axis (cf. Heading 1962, chap. 4). In our formalism, which essentially follows that of ray theory, transition points occur at singularities of (20): at the points where C_1 vanishes (real and imaginary parts). In our examples, C_1 does not vanish on the real axis, but it might vanish somewhere in the complex plane. However, since $A(X)$ does not become unduly large on the real axis, the transition points, if there are any, do not lie very close to the axis. Hence, l is expected to be $O(1)$, and the amplitude of any reflected wave would be very small.

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† In slowly varying flows that support neutral waves or in steady parallel flows that yield instability, it can be shown that the real part of the group velocity defines 'characteristics' in x, t space along which certain information is known; for example, the frequency and/or the wavenumber may remain constant along these lines. To the best of our knowledge, similar interpretations do not yet exist for flows that are both slowly varying *and* unstable.

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